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Supersymmetry and noncommutative geometry ★

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Abstract

The purpose of this article is to apply the concept of the spectral triple, the starting point for the analysis of noncommutative spaces in the sense of Connes (1994), to the case where the algebra $\mathcal{A}$ contains both bosonic and fermionic degrees of freedom. The operator $\mathcal{D}$ of the spectral triple under consideration is the square root of the Dirac operator and thus the forms of the generalized differential algebra constructed out of the spectral triple are in a representation of the Lorentz group with integer spin if the form degree is even and half-integer spin if the form degree is odd. However, we find that the 2-forms, obtained by squaring the connection, contain exactly the components of the vector multiplet representation of the supersymmetry algebra. This allows to construct an action for supersymmetric Yang–Mills theory in the framework of noncommutative geometry.

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1. Introduction

In the past years it has turned out that noncommutative geometry [1] offers a powerful mathematical framework for the study of fundamental interactions in physics. The construction of models for the electroweak and strong interaction in terms of noncommutative

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geometry, i.e. the Connes–Lott models [1,2] and the model of the Marseill–Mainz group [3–5], has led to a new qualitative insight into the spontaneous symmetry breaking mechanism of the Standard Model.

The basic idea of noncommutative geometry is to generalize geometric concepts such that they can be applied to more general situations where it is meaningless to consider, e.g. points connected by arcs. This allows to relax the physical notion of space–time such that our classical space–time may emerge as a "classical-limit" of a more general noncommutative space–time. This idea has been followed in [6] where uncertainty relations for space–time variables were implemented and their consequences were studied. A different approach to utilize this more general framework is to consider a sequence of finite-dimensional algebras which approximate the algebra of functions on a classical manifold, like the fuzzy sphere [7], and study a quantum field theory on such geometries [8–10]. Here the noncommutativity of the geometries serves as a regulator for the field theory.

However, the novel feature of these approaches to derive the Standard Model in the framework of noncommutative geometry is not a generalization of space–time itself. In those models a discrete space, i.e. a space consisting of two points, is added to a conventional space–time. This effects the internal symmetries of the theory such that the Higgs becomes a part of a generalized gauge potential. Similar ideas have been followed in [11]. It has turned out that the Connes–Lott models and their successors [12] based on real spectral triples do not only lead to qualitative restrictions compared to conventional Yang–Mills Higgs models, but also serve numerical relations for the Higgs mass and top-mass [13].

We take this as a motivation to explore the concept of the spectral triple, the basic input data for Connes–Lott models, in a more general context. In Connes–Lott models the notion of space–time was generalized to incorporate the symmetry breaking mechanism of internal symmetries. In this article we will analyze a spectral triple of a supermanifold, i.e. of a generalization of space–time which includes fermionic degrees of freedom. This leads to a generalization of space–time symmetry, namely to supersymmetry. The generalized differential algebra, which is constructed out of the spectral triple, is used to derive an action for $N = 1$ supersymmetric Yang–Mills theory although it is not the usual superdifferential algebra [14]. The basic difference of the generalized differential algebra to the conventional superdifferential algebra [14] is the absence of space–time differentials, i.e. the absence of space–time or vectorial 1-forms. Thus the differential algebra is generated only by spinorial 1-forms and the differential splits into a holomorphic and an antiholomorphic part, which are in $(\frac{1}{2}, 0)$, the left-handed spin $\frac{1}{2}$ representation, resp. in $(0, \frac{1}{2})$, the right-handed spin $\frac{1}{2}$ representation of the spin group. As in usual Yang–Mills theory, the action is obtained by squaring the curvature, which itself is the square of a covariant derivative.

There are several articles in the literature which deal with various aspects of supersymmetry and noncommutative geometry. For example in [15] noncommutative geometry is applied to supersymmetric constructive field theory. Furthermore let us just mention those articles which have a direct relation to models for the electroweak interaction. For the Marseill–Mainz model a $\mathbb{Z}_2$ graded structure plays a fundamental role in the sense that this model is based on a supergroup. However, the supergroup is not a symmetry group and therefore supersymmetry is not realized [5]. For the construction of gauge theories based on graded groups see e.g. [16,17].

The relation of supersymmetry and Connes–Lott models was investigated by Chamseddine [18] who explored the possibilities of arranging the elements of spectral triples of Connes–Lott models such that the resulting model is supersymmetric. Note, however, that although we also use the concept of spectral triples our work is essentially different from [18].

In Section 2 we briefly recall the definition of the spectral triple which allows us to indicate the starting point of our construction. Furthermore we introduce the commutative $*$-algebra $\mathcal{A}$ of the spectral triple which is a modified algebra of superfields. The construction of the spectral triple is completed by specifying the representation of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ and by defining the self-adjoint unbounded operator $\mathcal{D}$. In Section 3, after a brief outline of the general procedure, we start the construction of the generalized differential algebra with the definition of the generalized Clifford algebra. Its holomorphic structure and the relation to supersymmetry are discussed. The construction of the generalized differential algebra is completed in Section 4. A supersymmetric invariant inner product is constructed in Section 5. Section 6 contains our derivation of supersymmetric Yang–Mills theory. This article ends with some concluding remarks in Section 7.

2. The spectral triple

The basic object in noncommutative geometry defining the geometrical framework is the spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ [1,12]. $\mathcal{A}$, the first element of this triple, is an associative $*$-algebra of bounded operators with a unit in a Hilbert-space $\mathcal{H}$, the second element of the spectral triple. The last element, $\mathcal{D}$, is a self-adjoint unbounded operator in $\mathcal{H}$ such that:

(i) $\mathcal{D}$ has a compact inverse (modulo a finite-dimensional kernel),

(ii) $[\mathcal{D}, a] = \mathcal{D}a - a\mathcal{D}$ is bounded for any $a \in \mathcal{A}$.

Frequently the last two objects $(\mathcal{H}, \mathcal{D})$ are called a K-cycle over $\mathcal{A}$. These three elements together encode all geometric information of a space as spectral data. For example, it is possible to construct a differential algebra for this space, where the operator $\mathcal{D}$ defines the differential. This is the starting point for Yang–Mills theory in noncommutative geometry [1] (see, e.g. [19,20] for a review). We should mention that we only gave the definition of spectral triples of compact spaces. However, it is also possible to define spectral triples for spaces which are only locally compact [12].

The spectral triple describing the geometry of a compact spin manifold $\mathcal{M}$ is given by $(C^\infty(\mathcal{M}), L_2(S), D)$, where $C^\infty(\mathcal{M})$ are the smooth functions on $\mathcal{M}$, $L_2(S)$ is the Hilbert space of square-integrable spin-sections and D is the usual Dirac operator [1,12]. The differential algebra derived from this triple is the de Rham algebra of differential forms on $\mathcal{M}$.

Let us express the operator $\mathcal{D}$ of this example, the Dirac operator, in a somewhat different terms which refers to (symmetry) transformation of $\mathcal{M}$. Thus we think of the Dirac perator $D = \gamma^\mu \nabla_\mu$ as a composition of two kinds of objects:

(i) the generators of parallel displacement or covariant derivative ∇_μ,

(ii) the generators of the Clifford algebra corresponding to the vector space of generators of parallel displacement.

There is a well-known generalization of this Lie algebra of parallel displacement in physics: the supersymmetry (for the rest of the article we restrict ourselves to the case in which the manifold $\mathcal{M}$ is flat), which is generated by Q and $\overline{Q}$. The fundamental commutation relation is

$$[\varepsilon Q, \overline{\varepsilon}\overline{Q}] = 2\mathrm{i}\varepsilon\sigma^{\mu}\overline{\varepsilon}\partial_{\mu}, \tag{2.1}$$

where ε is a constant anticommuting chiral spinor and $\overline{\varepsilon}$ is a antichiral spinor, related to ε by charge conjugation, i.e. ε and $\overline{\varepsilon}$ together form a Majorana spinor. This implies that in four dimensions an Euclidean space–time metric is excluded. However, the noncommutative analog of an integral, the Dixmier trace, is defined only on Euclidean space. On the other hand, the construction of the generalized differential algebra does not refer to the signature of space–time. Furthermore, the special structure of the Hilbert space, which will be defined below, allows us to define an inner product on the generalized Clifford algebra which induces an inner product on the generalized differential forms. This leads to a supersymmetric invariant action which is defined without using the Dixmier trace.

The purpose of this article is to encode the generalization of space–time in the sense of Eq. (2.1) in the spectral triple. Therefore we have to extend the algebra of (bosonic) functions, $C^{\infty}(\mathcal{M})$ by fermionic quantities. Thus we have to include the spinors as anticommuting, i.e. as Grassmann-odd, objects in the algebra. In order to maintain the regularity of the algebra we restrict ourselves to the dense subspace of smooth spinors $\Gamma(S) \subset L_2(S)$. Furthermore it is useful to split $\Gamma(S)$ into its irreducible parts of the Lorentz group:

$$\Gamma(S) = \Gamma(S_+) \oplus \Gamma(S_-), \qquad \Psi = (\psi_\alpha, \overline{\chi}^{\dot\alpha}). \tag{2.2}$$

The indices $\alpha \in \{1,2\}$ and $\dot\alpha \in \{1,2\}$ can be raised and lowered with the antisymmetric tensors $\varepsilon^{\alpha\beta}$ and $\varepsilon_{\dot\alpha\dot\beta}$. We use the convention of [14], i.e.

$$\varepsilon_{21} = \varepsilon^{12} = 1 \tag{2.3}$$

for both ε-tensors with dotted and undotted indices.

The multiplication rules of spinors are most conveniently described with the help of a constant anticommuting Majorana spinor θ. Thus the algebra $\mathcal{A}_0$ is generated by elements g_0 of the form

$$g_0 = f + \theta^\alpha\psi_\alpha + \overline{\theta}_{\dot\alpha}\overline{\chi}^{\dot\alpha}, \quad f \in C^{\infty}(\mathcal{M}),\ (\psi_\alpha, \overline{\chi}^{\dot\alpha}) \in \Gamma(S), \tag{2.4}$$

is the usual algebra of superfields [14]. A general element $a_0 \in \mathcal{A}_0$ can be expanded in powers of $(\theta, \overline{\theta})$ as follows:

$$\begin{aligned} a_0 = {}& a_1 + \theta^\alpha a_{2\alpha} + \overline{\theta}_{\dot\alpha} a_3{}^{\dot\alpha} + \theta^\alpha\overline{\theta}_{\dot\alpha} a_{4\alpha}^{\dot\alpha} + \theta\theta a_5 + \overline{\theta}\overline{\theta} a_6 \\ & + \overline{\theta}\overline{\theta}\theta^\alpha a_{7\alpha} + \theta\theta\overline{\theta}_{\dot\alpha} a_8{}^{\dot\alpha} + \theta\theta\overline{\theta}\overline{\theta} a_9\ . \end{aligned} \tag{2.5}$$

The $*$-operation is defined on the generators g_0 as complex conjugation on functions and charge conjugation on spinors. This definition extends uniquely to the whole algebra $\mathcal{A}_0$.

However, this algebra is not well suited for our purpose as will become clear when we compute the generalized differential algebra. Therefore we enlarge the algebra by taking spinor doublets as generators of the algebra $\mathcal{A}$, i.e., we define $\mathcal{A}$ to be the algebra which is generated by elements g of the following form

$$g = f + \theta^\alpha \psi_\alpha \otimes v + \overline{\theta}_{\dot\alpha} \overline{\chi}^{\dot\alpha} \otimes \overline{w},$$
$$f \in C^\infty(\mathcal{M}), \quad (\psi_\alpha, \overline{\chi}^{\dot\alpha}) \in \Gamma(S), \quad v, \overline{w} \in \mathbb{C}^2. \tag{2.6}$$

For a generator g as in Eq. (2.6) the $*$-operation is defined to be

$$g^* = \overline{f} + \theta^\alpha \chi_\alpha \otimes w + \overline{\theta}_{\dot\alpha} \overline{\psi}^{\dot\alpha} \otimes \overline{v}\,. \tag{2.7}$$

The multiplication for elements in $\mathbb{C}^2$ is just the totally symmetrized tensor-multiplication. Thus a general element $a \in \mathcal{A}$ can be expanded in powers of $(\theta, \overline{\theta})$ as follows:

$$\begin{aligned}a = {}& a_1 + \theta^\alpha a_{2\alpha} \otimes v^{(2)} + \overline{\theta}_{\dot\alpha} a_3{}^{\dot\alpha} \otimes v^{(3)} \\ &+ \theta^\alpha \overline{\theta}_{\dot\alpha} a_4{}^{\dot\alpha}_\alpha \otimes (v_1^{(4)} \otimes_s v_2^{(4)}) + \theta\theta a_5 \otimes (v_1^{(5)} \otimes_s v_2^{(5)}) + \overline{\theta\theta} a_6 \otimes (v_1^{(6)} \otimes_s v_2^{(6)}) \\ &+ \overline{\theta\theta}\theta^\alpha a_{7\alpha} \otimes (v_1^{(7)} \otimes_s v_2^{(7)} \otimes_s v_3^{(7)}) + \theta\theta\overline{\theta}_{\dot\alpha} a_8{}^{\dot\alpha} \otimes (v_1^{(8)} \otimes_s v_2^{(8)} \otimes_s v_3^{(8)}) \\ &+ \theta\theta\overline{\theta\theta} a_9 \otimes (v_1^{(9)} \otimes_s v_2^{(9)} \otimes_s v_3^{(9)} \otimes_s v_4^{(9)}).\end{aligned} \tag{2.8}$$

There is no direct definition of supersymmetry generators on $\mathcal{A}$ which could be obtained as generalization of the supersymmetry generators on $\mathcal{A}_0$ which are defined as follows:

$$Q_\alpha = \partial_\alpha - \mathrm{i}\,\partial\!\!\!/_{\alpha\dot\alpha} \overline{\theta}^{\dot\alpha}, \quad \overline{Q}_{\dot\alpha} = -\overline{\partial}_{\dot\alpha} + \mathrm{i}\theta^\alpha \partial\!\!\!/_{\alpha\dot\alpha}. \tag{2.9}$$

Here $\partial_\alpha = \partial/\partial\theta^\alpha$, $\overline{\partial}_{\dot\alpha} = \partial/\partial\overline{\theta}^{\dot\alpha}$ denote the derivatives with respect to θ^α and $\overline{\theta}_{\dot\alpha}$ and $\partial\!\!\!/_{\alpha\dot\alpha} = \sigma^\mu_{\alpha\dot\alpha}\partial_\mu$, where $\sigma^i = -\overline{\sigma}^i$, $i = 1, 2, 3$, denote the Pauli matrices and $\sigma^0 = \overline{\sigma}^0 = 1_{2\times 2}$.

However, for any fixed $v \in \mathbb{C}^2$ with $\overline{v} = v$ there is an embedding

$$i_v : \mathcal{A}_0 \longrightarrow \mathcal{A}, \tag{2.10}$$

which is defined on the generators of $\mathcal{A}_0$ as

$$i_v(f + \theta^\alpha \psi_\alpha + \overline{\theta}_{\dot\alpha} \overline{\chi}^{\dot\alpha}) = (f + \theta^\alpha \psi_\alpha \otimes v + \overline{\theta}_{\dot\alpha} \overline{\chi}^{\dot\alpha} \otimes v)$$
$$\forall f \in C^\infty(\mathcal{M}) \quad \forall (\psi_\alpha, \overline{\chi}^{\dot\alpha}) \in \Gamma(S) \tag{2.11}$$

Thus this allows us to define the supersymmetry generators on the subalgebra $\mathcal{A}_v = i_v(\mathcal{A}_0)$ of $\mathcal{A}$ as

$$Q_\alpha^{(v)} = i_v Q_\alpha i_v^{-1}, \qquad \overline{Q}_{\dot\alpha}^{(v)} = i_v \overline{Q}_{\dot\alpha} i_v^{-1}. \tag{2.12}$$

Explicitly these generators read

$$Q_\alpha^{(v)} = \partial_\alpha \otimes v^* - \mathrm{i}\,\partial\!\!\!/_{\alpha\dot\alpha} \overline{\theta}^{\dot\alpha} \otimes v, \qquad \overline{Q}_{\dot\alpha}^{(v)} = -\overline{\partial}_{\dot\alpha} \otimes v^* + \mathrm{i}\theta^\alpha \partial\!\!\!/_{\alpha\dot\alpha} \otimes v, \tag{2.13}$$

where v^* denotes the dual vector of v, i.e. $v^*(v) = 1$. However, the action of v^* on higher powers of v is defined to be

$$v^*(v^n) = v^*(v \otimes_s \cdots \otimes_s v) = v^{n-1}. \tag{2.14}$$

This definition follows directly from Eq. (2.12). The action of v^* on higher powers of v differs from the action of a derivative and therefore there is no direct extension of this definition to symmetric tensor products of arbitrary vectors. On the other hand, if v^* would act like a derivative on tensor products, the operators defined in Eq. (2.13) would not generate a supersymmetry algebra.

We now turn to the next element of the spectral triple, the space $\mathcal{H}$, which carries a representation of $\mathcal{A}$. A representation space $\mathcal{H}_\pi$ can be constructed out of the algebra $\mathcal{A}_0$ generated by elements of the form as in Eq. (2.4) as follows:

$$\mathcal{H}_\pi = \mathcal{A}_0 \otimes \mathbb{Z} = \bigoplus_{n\in\mathbb{Z}} \mathcal{A}_n, \tag{2.15}$$

where we have defined

$$\mathcal{A}_n = \mathcal{A}_0 \otimes n \quad \forall n \in \mathbb{Z}. \tag{2.16}$$

Thus $\mathcal{H}_\pi$ is the $\mathbb{Z}$-fold copy of $\mathcal{A}_0$. We will call the index n the S-number, i.e. for all elements $\Phi_n \in \mathcal{A}_n$ it is

$$S(\Phi_n) = n. \tag{2.17}$$

The inner product $(\cdot,\cdot)$ on this space we define for any $\Psi_k = \Psi \otimes k \in \mathcal{A}_k$, $\Phi_l = \Phi \otimes l \in \mathcal{A}_l$ as

$$(\Psi_k, \Phi_l) = \delta_{k+l,0} \int\limits_{\mathcal{M}} \mathrm{d}V (\Psi^* \Phi)|_{\theta\theta\overline{\theta\theta}}, \tag{2.18}$$

where $|_{\theta\theta\overline{\theta\theta}}$ denotes the projection onto the $\theta\theta\overline{\theta\theta}$-component of the $\theta, \overline{\theta}$-expansion of the superfields. Thus it is the usual indefinite inner product on the algebra of superfields $\mathcal{A}_0$ multiplied by an indefinite inner product on $\mathbb{Z}$. Note that the supersymmetry generators, as defined in Eq. (2.9), also are well defined on $\mathcal{H}_\pi$ and that the inner product, defined in Eq. (2.18), is invariant under supersymmetry transformations.

For the definition of a representation of $\mathcal{A}$ on $\mathcal{H}_\pi$ we have to introduce two operators S_+ and S_- which act on elements $\Psi \otimes k \in \mathcal{H}_\pi$ as follows:

$$S_+(\Psi \otimes k) = \Psi \otimes (k+1), \quad S_-(\Psi \otimes k) = \Psi \otimes (k-1), \qquad \forall k \in \mathbb{Z}. \tag{2.19}$$

These S_+ and S_- are self-adjoint and $S_+S_- = S_-S_+ = 1$. With these operators we can define a representation $\pi(a)$ on $\mathcal{H}_\pi$ for all $a \in \mathcal{A}$ with a $\theta, \overline{\theta}$-expansion as in Eq. (2.8) as follows:

$$\begin{aligned}\pi(a) = {} & a_1 + \theta^\alpha a_{2\alpha} \otimes V_1^{(2)} + \overline{\theta}_{\dot\alpha} a_3{}^{\dot\alpha} \otimes V_1^{(3)} \\ & + \theta^\alpha \overline{\theta}_{\dot\alpha} a_4{}_\alpha^{\dot\alpha} \otimes V_1^{(4)} V_2^{(4)} + \theta\theta a_5 \otimes V_1^{(5)} V_2^{(5)} + \overline{\theta\theta} a_6 \otimes V_1^{(6)} V_2^{(6)}\end{aligned}$$

$$+\overline{\theta\theta}\theta^{\alpha}a_{7\alpha}\otimes V_1^{(7)}V_2^{(7)}V_3^{(7)}+\theta\theta\overline{\theta}_{\dot\alpha}a_8{}^{\dot\alpha}\otimes V_1^{(8)}V_2^{(8)}V_3^{(8)}$$
$$+\theta\theta\overline{\theta\theta}a_9\otimes V_1^{(9)}V_2^{(9)}V_3^{(9)}V_4^{(9)}, \tag{2.20}$$

where we used the notation

$$V_i^{(j)}=(v_i^{(j)})_+S_++(v_i^{(j)})_-S_-,\quad v_i^{(j)}=((v_i^{(j)})_+,(v_i^{(j)})_-)\in\mathbb{C}^2. \tag{2.21}$$

From Eq. (2.20) we can read off the range of the S-numbers of the components in the $\theta,\overline{\theta}$ expansion

$$S(a_k)\in\{-n,-n+2,\dots,n-2,n\}, \tag{2.22}$$

where $n\le 4$ is the power of $(\theta,\overline{\theta})$ at which the component appears.

Due to the fact that $\mathcal{A}_0$ has a unit element 1, there is an invariant subspace $\mathcal{H}_{\mathcal{A}}\subset\mathcal{H}_\pi$ which is generated by $\mathcal{A}$,

$$\mathcal{H}_{\mathcal{A}}=\mathcal{A}(1\otimes 0)=\mathcal{A}|0\rangle. \tag{2.23}$$

This allows us to define an (indefinite) inner product on $\mathcal{A}$ as

$$\langle a,b\rangle=((a|0\rangle),(b|0\rangle))=\langle 0|a^*b|0\rangle\quad\forall a,b\in\mathcal{A}. \tag{2.24}$$

Note, that this inner product is degenerate for the components a_7, a_8 and a_9 with the following S-numbers:

$$S(a_7)=S(a_8)=\pm 3,\qquad S(a_9)=\pm 2,\pm 4. \tag{2.25}$$

Eq. (2.24) induces also an inner product on $\mathcal{A}_v$ which depends on $v=(v_+,v_-)\in\mathbb{C}^2$ and can be completely degenerate

$$\langle a_v,b_v\rangle=\langle\pi\circ i_v(a_0),\pi\circ i_v(b_0)\rangle=6v_+^2v_-^2(a_0,b_0)\quad\forall a_0,b_0\in\mathcal{A}_0. \tag{2.26}$$

Thus we are led to require

$$v_+\neq 0,\quad v_-\neq 0. \tag{2.27}$$

We now turn to the last element of the spectral triple, the unbounded self-adjoint operator $\mathcal{D}$. We construct this operator out of the two operators D_α and $\overline{D}_{\dot\alpha}$

$$D_\alpha=\partial_\alpha+\mathrm{i}\,\not{\partial}_{\alpha\dot\alpha}\overline{\theta}^{\dot\alpha},\quad \overline{D}_{\dot\alpha}=-\overline{\partial}_{\dot\alpha}-\mathrm{i}\theta^\alpha\not{\partial}_{\alpha\dot\alpha}, \tag{2.28}$$

which are associated to the supersymmetry generators as defined in Eq. (2.9). By construction they enjoy the property that all anticommutators of these operators with the supersymmetry generators vanish and the only nonvanishing anticommutator is

$$[D_\alpha,\overline{D}_{\dot\alpha}]_+=-2\mathrm{i}\,\not{\partial}_{\alpha\dot\alpha}. \tag{2.29}$$

Furthermore, we can use them to define the following subalgebras of $\mathcal{A}$:

$$\mathcal{A}_+=\{a\in\mathcal{A}|\overline{D}_{\dot\alpha}a=0\},\qquad \mathcal{A}_-=\{a\in\mathcal{A}|D_\alpha a=0\}. \tag{2.30}$$

The algebra $\mathcal{A}$ can be generated with the two algebras $\mathcal{A}_+$ and $\mathcal{A}_-$, i.e. any element $a \in \mathcal{A}$ can be written as

$$a = \sum_i a_+^{(i)} a_-^{(i)}, \quad a_+^{(i)} \in \mathcal{A}_+,\ a_-^{(i)} \in \mathcal{A}_-. \tag{2.31}$$

This fact will turn out to be important for the complex structure of the super-Clifford algebra and it will be very useful in the computation of $\Omega_{\mathcal{D}}\mathcal{A}$.

We only use the two operators for the construction of $\mathcal{D}$ since the space–time derivatives are already encoded in D and $\overline{D}$. In other words, the operator $\mathcal{D}$ is not constructed out of the full set of operators which form a basis of the supersymmetry algebra as a vector space. $\mathcal{D}$ contains only the generating operators from which the complete algebra is obtained via commutation relations. Thus the operator $\mathcal{D}$ constructed out of D and $\overline{D}$ has a natural interpretation as a square root of the Dirac operator. As a consequence the 1-forms of the resulting differential algebra will be in the spin $\frac{1}{2}$ representation of the Lorentz group.

Having fixed the derivative part of $\mathcal{D}$ we still have to construct the "Clifford algebra" part. However, since the operators D and $\overline{D}$ are odd, i.e. they obey anticommutation relations the corresponding "Clifford algebra" has to fulfill the following commutation relations:

$$\eta^\alpha \eta^\beta - \eta^\beta \eta^\alpha = 2\mathrm{i}\varepsilon^{\alpha\beta}, \quad \overline{\eta}_{\dot\alpha}\overline{\eta}_{\dot\beta} - \overline{\eta}_{\dot\beta}\overline{\eta}_{\dot\alpha} = 2\mathrm{i}\varepsilon_{\dot\alpha\dot\beta}, \quad \eta^\alpha\overline{\eta}_{\dot\beta} - \overline{\eta}_{\dot\beta}\eta^\alpha = 0, \tag{2.32}$$

where the right-hand sides are dictated by the symplectic form which defines the inner product on spinors. The η^α and $\overline{\eta}_{\dot\alpha}$ have to be related by hermitean conjugation since the operator $\mathcal{D}$ has to be self-adjoint. Thus Eq. (2.32) defines a Heisenberg algebra which has a unitary representation on $\mathcal{H}_{\mathrm{H}} = L_2(\mathbb{C} \oplus \overline{\mathbb{C}})$.

The total space $\mathcal{H}$ of the spectral triple is the tensor product of the representation space of $\mathcal{A}$ and the representation space of the Heisenberg algebra

$$\mathcal{H} = \mathcal{H}_{\mathrm{H}} \otimes \mathcal{H}_{\mathcal{A}} \tag{2.33}$$

and the operator $\mathcal{D}$ is defined on this space as

$$\mathcal{D} = \eta^\alpha \otimes D_\alpha + \overline{\eta}_{\dot\alpha} \otimes \overline{D}^{\dot\alpha}. \tag{2.34}$$

Unless there is no risk of confusion we drop the tensor notation and simply write $\pi(a) = a$ and $\mathcal{D} = \eta^\alpha D_\alpha + \overline{\eta}_{\dot\alpha}\overline{D}^{\dot\alpha}$.

3. The universal differential envelope and the super-Clifford algebra

Let us start this section with a brief description of the general construction of a generalized differential algebra in noncommutative geometry [1] (for detailed reviews the reader may consult [19,20]).

The first step is to construct the universal differential envelope $\Omega\mathcal{A}$ by associating to each element $a \in \mathcal{A}$ the symbol δa. $\Omega\mathcal{A}$ is the free algebra generated by the symbols $a, \delta b$ with $a, b \in \mathcal{A}$, modulo the relation

$$\delta(ab) = \delta a\, b + a\delta b. \tag{3.1}$$

With the definition

$$\begin{aligned} \delta(a_0\delta a_1\cdots\delta a_k) &:= \delta a_0\,\delta a_1\cdots\delta a_k, \\ \delta(\delta a_1\cdots\delta a_k) &:= 0, \end{aligned} \tag{3.2}$$

$\Omega\mathcal{A}$ becomes a $\mathbb{N}$-graded differential algebra with the odd differential δ, $\delta^2 = 0$. By defining

$$\delta(a)^* = -\delta(a^*) \tag{3.3}$$

the $*$-operation is extended uniquely to $\Omega\mathcal{A}$.

The next step is to extend the representation π of $\mathcal{A}$ to a representation $\pi_{\mathcal{D}}$ of $\Omega\mathcal{A}$. Since $[\mathcal{D}, a]$ is bounded for any $a \in \mathcal{A}$ we can define for all $k \in \mathbb{N}$

$$\begin{aligned} \pi_{\mathcal{D}} : \Omega^k\mathcal{A} &\longrightarrow \mathcal{B}(\mathcal{H}) \\ \pi_{\mathcal{D}}(a_0\delta a_1\cdots\delta a_k) &= a_0[\mathcal{D}, a_1]\cdots[\mathcal{D}, a_k]. \end{aligned} \tag{3.4}$$

Although $\pi_{\mathcal{D}}$ is a representation of the algebra $\Omega\mathcal{A}$ it fails to be a homomorphism of differential algebras. The trouble is that from

$$\pi_{\mathcal{D}}(\omega) = 0, \quad \omega \in \Omega\mathcal{A}, \tag{3.5}$$

it does not follow that

$$\pi_{\mathcal{D}}(\delta\omega) = 0. \tag{3.6}$$

Furthermore, there is no natural grading on $\ker(\pi_{\mathcal{D}})$ in general. To obtain a graded differential algebra one has to identify these disturbing elements which form a graded differential ideal $\mathcal{J}$. This ideal is given as [1]

$$\begin{aligned} \mathcal{J}^n &= (\ker\pi_{\mathcal{D}} \cap \Omega^n\mathcal{A}) \cup \delta(\ker\pi_{\mathcal{D}} \cap \Omega^{n-1}\mathcal{A}), \\ \mathcal{J} &= \bigoplus_{n\in\mathbb{N}} \mathcal{J}^n. \end{aligned} \tag{3.7}$$

Finally, the generalized differential algebra $\Omega_{\mathcal{D}}\mathcal{A}$ is defined as the following quotient algebra

$$\Omega^n_{\mathcal{D}}\mathcal{A} = \frac{\Omega^n\mathcal{A}}{\mathcal{J}^n}, \qquad \Omega_{\mathcal{D}}\mathcal{A} = \bigoplus_{n\in\mathbb{N}} \Omega^n_{\mathcal{D}}. \tag{3.8}$$

However, before we start to compute $\mathcal{J}$ and $\Omega_{\mathcal{D}}\mathcal{A}$ let us first discuss the representation $\pi_{\mathcal{D}}$ of $\Omega\mathcal{A}$. For the spectral triple $(C^\infty(\mathcal{M}), L_2(S), \partial\!\!\!/)$ the image $\pi_{\partial\!\!\!/}(\Omega\mathcal{A})$ is the Clifford bundle $Cl(\mathcal{M})$ over $\mathcal{M}$ [1]. Thus in our case we call the image of $\pi_{\mathcal{D}}$, $Cl_{\mathcal{D}}\mathcal{A} = \pi_{\mathcal{D}}(\Omega\mathcal{A})$, the super-Clifford algebra of the spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$. From the fact that $\mathcal{A}$ is generated by chiral and antichiral superfields and

$$[\eta^\alpha D_\alpha, a][\bar\eta_{\dot\alpha}\overline{D}^{\dot\alpha}, b] + [\bar\eta_{\dot\alpha}\overline{D}^{\dot\alpha}, b][\eta^\alpha D_\alpha, a] = 0 \quad \forall a, b \in \mathcal{A}, \tag{3.9}$$

we conclude that

$$\pi_{\mathcal{D}}(\Omega^1\mathcal{A}) = Cl^{(1,0)}_{\mathcal{D}}\mathcal{A} \oplus Cl^{(0,1)}_{\mathcal{D}}\mathcal{A}, \tag{3.10}$$

where $Cl_{\mathcal{D}}^{(1,0)}\mathcal{A}$ and $Cl_{\mathcal{D}}^{(0,1)}\mathcal{A}$ are linear spaces defined as

$$
\begin{aligned}
Cl_{\mathcal{D}}^{(1,0)}\mathcal{A} &= \left\{\pi_{\mathcal{D}}\left(\sum_i a^{(i)}\delta b_+^{(i)}\right) = \sum_i a^{(i)}\eta^\alpha D_\alpha b_+^{(i)} | a^{(i)} \in \mathcal{A},\ b_+^{(i)} \in \mathcal{A}_+\right\},\\
Cl_{\mathcal{D}}^{(0,1)}\mathcal{A} &= \left\{\pi_{\mathcal{D}}\left(\sum_i a^{(i)}\delta b_-^{(i)}\right) = \sum_i a^{(i)}\overline{\eta}_{\dot\alpha}\overline{D}^{\dot\alpha} b_-^{(i)} | a^{(i)} \in \mathcal{A},\ b_-^{(i)} \in \mathcal{A}_-\right\}.
\end{aligned}
\tag{3.11}
$$

The algebra $\Omega\mathcal{A}$ is generated by 1-forms $a\delta b \in \Omega^1\mathcal{A}$ therefore the decomposition (3.10) extends to the images of higher forms

$$
\pi_{\mathcal{D}}(\Omega^k\mathcal{A}) = \bigoplus_{l=0}^{k} Cl_{\mathcal{D}}^{(k-l,l)}\mathcal{A}. \tag{3.12}
$$

A generic element $v \in Cl_{\mathcal{D}}^{(k,l)}\mathcal{A}$ is of the form

$$
\begin{aligned}
v &= \eta^{\alpha_1}\cdots\eta^{\alpha_k}\overline{\eta}_{\dot\alpha_1}\cdots\overline{\eta}_{\dot\alpha_l} v_{\alpha_1\cdots\alpha_k}^{\dot\alpha_1\cdots\dot\alpha_l}\\
&= \eta^{\alpha_1}\cdots\eta^{\alpha_k}\overline{\eta}_{\dot\alpha_1}\cdots\overline{\eta}_{\dot\alpha_l}\\
&\quad\times\left(\sum_i a^{(i)}[D_{\alpha_1}, b_1^{(i)}]\cdots[D_{\alpha_k}, b_k^{(i)}][\overline{D}^{\dot\alpha_1}, c_1^{(i)}]\cdots[\overline{D}^{\dot\alpha_l}, c_l^{(i)}]\right)
\end{aligned}
\tag{3.13}
$$

with $a^{(i)} \in \mathcal{A}, b^{(i)} \in \mathcal{A}_+$ and $c^{(i)} \in \mathcal{A}_-$. Thus the elements of $Cl_{\mathcal{D}}^{(k,l)}\mathcal{A}$ are tensor superfields with k holomorphic and l antiholomorphic spinor indices.

Let us now turn to the S-numbers of the elements in $Cl_{\mathcal{D}}\mathcal{A}$. The ∂_α, resp. $\overline{\partial}_{\dot\alpha}$ part of the operator D_α, resp. $\overline{D}_{\dot\alpha}$ shifts the coefficients of higher powers of $\theta, \overline{\theta}$ to lower powers and thus also the number of $S_\pm$ operators (which coincides with the power of $\theta, \overline{\theta}$ for elements in $\mathcal{A}$) is shifted to lower powers of θ, $\overline{\theta}$. Thus for any element $\omega \in Cl_{\mathcal{D}}^{(k,l)}\mathcal{A}$ with a θ, $\overline{\theta}$-expansion as in (2.20) the range for the S-numbers of the coefficients of $(\theta, \overline{\theta})^n$ is

$$
S(\omega_j) = \{-(k+l+n), -(k+l+n)+2, \ldots, k+l+n-2, k+l+n\}, \tag{3.14}
$$

where we suppressed the explicit dependence of η, $\overline{\eta}$ and $S_\pm$.

Again, for any real $v \in \mathbb{C}^2$ there is a subalgebra of $Cl_{\mathcal{D}}\mathcal{A}$ on which there is a well-defined action of supersymmetry. Thus there is an extension of i_v as defined in Eq. (2.11) which embeds $\mathcal{A}_0^{(k,l)}$, i.e. tensor superfields with k holomorphic and l antiholomorphic indices, into $Cl_{\mathcal{D}}^{(k,l)}\mathcal{A}$. We define this embedding $i_{v^{k+l}}$ on the components of the $(\theta, \overline{\theta})$-expansion as

$$
i_{v^{k+l}}((\theta, \overline{\theta})^n\omega_{(n)}) = (\theta, \overline{\theta})^n\omega_{(n)}V^{n+k+l}. \tag{3.15}
$$

However, note that $(Cl_{\mathcal{D}}\mathcal{A})_{v^{k+l}} = i_{v^{k+l}}(\mathcal{A}_0^{(k,l)})$ is not invariant under the action of $\mathcal{D}$:

$$
\begin{aligned}
&[\mathrm{i}\eta^\alpha \partial\!\!\!/_{\alpha\dot\alpha}\overline{\theta}^{\dot\alpha}, (Cl_{\mathcal{D}}^{(k,l)}\mathcal{A})_{v^{k+l}}] \not\subset (Cl_{\mathcal{D}}^{(k+1,l)}\mathcal{A})_{v^{k+l+1}},\\
&[\mathrm{i}\overline{\eta}_{\dot\alpha}\overline{\partial\!\!\!/}^{\dot\alpha\alpha}\theta_\alpha, (Cl_{\mathcal{D}}^{(k,l)}\mathcal{A})_{v^{k+l}}] \not\subset (Cl_{\mathcal{D}}^{(k,l+1)}\mathcal{A})_{v^{k+l+1}}
\end{aligned}
\tag{3.16}
$$

because these parts of $\mathcal{D}$ proportional to θ resp. $\overline{\theta}$ which causes a shift of components of lower powers of $(\theta, \overline{\theta})$ to higher powers of $(\theta, \overline{\theta})$ whereas the power of V remains unchanged.

However, the embedding $i_{v^{k+l}}$ defined in Eq. (3.15) can be generalized in the following way:

$$i_{v^{k+l-2m}}((\theta, \overline{\theta})^n \omega_{(n)}) = (\theta, \overline{\theta})^n \omega_{(n)} V^{n+k+l-2m}, \quad 0 \leq 2m \leq k+l. \tag{3.17}$$

For all $k, l \in \mathbb{N}$ and $2m \leq k+l$ this defines a series of subspaces $(Cl_{\mathcal{D}}^{(k,l)}\mathcal{A})_{v^{k+l-2m}} \subset Cl_{\mathcal{D}}^{(k,l)}\mathcal{A}$ which carry a representation of the supersymmetry algebra. It is easy to check that these subspaces form a subalgebra of $Cl_{\mathcal{D}}\mathcal{A}$

$$(Cl_{\mathcal{D}}^{(k,l)}\mathcal{A})_{v^{k+l-2m}} \cdot (Cl_{\mathcal{D}}^{(r,s)}\mathcal{A})_{v^{r+s-2n}} = (Cl_{\mathcal{D}}^{(k+r,l+s)}\mathcal{A})_{v^{k+r+l+s-2(m+n)}} \tag{3.18}$$

and we define

$$(Cl_{\mathcal{D}}\mathcal{A})_v = \bigoplus_{k,l \in \mathbb{N}} \bigoplus_{m=0}^{2m=k+l} (Cl_{\mathcal{D}}^{(k,l)}\mathcal{A})_{v^{k+l-2m}}. \tag{3.19}$$

Note that it is

$$[\mathcal{D}, (Cl_{\mathcal{D}}\mathcal{A})_v] \subset (Cl_{\mathcal{D}}\mathcal{A})_v. \tag{3.20}$$

4. The generalized differential algebra $\Omega_{\mathcal{D}}\mathcal{A}$

Now we turn to the computation of $\mathcal{J}$ resp. $\mathcal{J}_{\mathcal{D}} = \pi_{\mathcal{D}}(\mathcal{J})$. The decomposition of $\pi_{\mathcal{D}}(\Omega^k\mathcal{A})$ in Eq. (3.12) induces the decomposition

$$\mathcal{J}_{\mathcal{D}}^k = \bigoplus_{l=0}^{k} \mathcal{J}_{\mathcal{D}}^{(k-l,l)} \tag{4.1}$$

and also

$$\Omega_{\mathcal{D}}^k\mathcal{A} = \bigoplus_{l=0}^{k} \Omega_{\mathcal{D}}^{(k-l,l)}\mathcal{A} = \bigoplus_{l=0}^{k} \frac{Cl_{\mathcal{D}}^{(k-l,l)}\mathcal{A}}{\mathcal{J}_{\mathcal{D}}^{(k-l,l)}}. \tag{4.2}$$

Since $\pi_{\mathcal{D}}(\mathcal{J}^1) = \{0\}$ the first nontrivial contributions to $\mathcal{J}_{\mathcal{D}}$ appears at the level of 2-forms, which splits into three parts

$$\pi_{\mathcal{D}}(\mathcal{J}^2) = \mathcal{J}_{\mathcal{D}}^{(2,0)} \oplus \mathcal{J}_{\mathcal{D}}^{(0,2)} \oplus \mathcal{J}_{\mathcal{D}}^{(1,1)}. \tag{4.3}$$

The first two spaces on the right hand side of Eq. (4.3), i.e. the holomorphic and antiholomorphic part of $\mathcal{J}_{\mathcal{D}}^2$ are determined by the following line of arguments: for any $a, b \in \mathcal{A}$ we set

$$\omega = -\delta(ab) + a\delta b + b\delta a \in \Omega^1\mathcal{A} \;\Rightarrow\; \pi_{\mathcal{D}}(\omega) = 0.$$

Thus it is $\delta\omega \in \mathcal{J}^2$ and we compute

$$\begin{aligned}\pi_{\mathcal{D}}(\delta\omega) &= \eta^\alpha\eta^\beta([D_\alpha, a][D_\beta, b] + [D_\alpha, b][D_\beta, a]) \\ &\quad +\overline{\eta}_{\dot\alpha}\overline{\eta}_{\dot\beta}([\overline{D}^{\dot\alpha}, a][\overline{D}^{\dot\beta}, b] + [\overline{D}^{\dot\alpha}, b][\overline{D}^{\dot\beta}, a]) \\ &= 2i\varepsilon^{\alpha\beta}[D_\alpha, a][D_\beta, b] + 2i\varepsilon_{\dot\alpha\dot\beta}[\overline{D}^{\dot\alpha}, a][\overline{D}^{\dot\beta}, b].\end{aligned} \tag{4.4}$$

From this we conclude that the holomorphic part $\mathcal{J}_{\mathcal{D}}^{(2,0)}$ contains all antisymmetric tensor superfields, i.e.

$$\mathcal{J}_{\mathcal{D}}^{(2,0)} = \{\eta^\alpha\eta^\beta w_{\alpha\beta} \in Cl_{\mathcal{D}}^{(2,0)}\mathcal{A} | w_{\alpha\beta} = -w_{\beta\alpha}\} \tag{4.5}$$

and also for the antiholomorphic part we find

$$\mathcal{J}_{\mathcal{D}}^{(0,2)} = \{\overline{\eta}_{\dot\alpha}\overline{\eta}_{\dot\beta} w^{\dot\alpha\dot\beta} \in Cl_{\mathcal{D}}^{(0,2)}\mathcal{A} | w^{\dot\alpha\dot\beta} = -w^{\dot\beta\dot\alpha}\}. \tag{4.6}$$

$\mathcal{J}_{\mathcal{D}}^{(2,0)}$ already generates the complete holomorphic part of $\mathcal{J}_{\mathcal{D}}$, i.e. for any $k \in \mathbb{N}$, $k \geq 2$ it is

$$\mathcal{J}_{\mathcal{D}}^{(k,0)} = \bigcup_{k=0}^{k-2} Cl_{\mathcal{D}}^{(k-l-2,0)}\mathcal{A}\mathcal{J}_{\mathcal{D}}^{(2,0)}Cl_{\mathcal{D}}^{(l,0)}\mathcal{A}. \tag{4.7}$$

To see that this holomorphic ideal in $Cl_{\mathcal{D}}^{(\bullet,0)}\mathcal{A}$ is the correct ideal it is sufficient to show that the holomorphic algebra $\Omega_{\mathcal{D}}^h\mathcal{H}$ with

$$\Omega_{\mathcal{D}}^h\mathcal{A} = \bigoplus_{n\in\mathbb{N}} \Omega_{\mathcal{D}}^{(n,0)}\mathcal{A} = \bigoplus_{n\in\mathbb{N}} \frac{Cl_{\mathcal{D}}^{(n,0)}\mathcal{A}}{\mathcal{J}_{\mathcal{D}}^{(n,0)}} \tag{4.8}$$

is a differential algebra. The algebra defined in Eq. (4.8) contains only totally symmetric tensor superfields, i.e. it is

$$\Omega_{\mathcal{D}}^{(k,0)}\mathcal{A} = \left\{\eta^{\alpha_1}\cdots\eta^{\alpha_k} w_{\alpha_1\cdots\alpha_k} \in Cl_{\mathcal{D}}^{(k,0)}\mathcal{A} | w_{\alpha_1\cdots\alpha_k} = \frac{1}{k!} w_{(\alpha_1\cdots\alpha_k)}\right\}, \tag{4.9}$$

where $(\alpha_1\cdots\alpha_k)$ denotes the sum over all permutations of the enclosed indices. The map $\sigma_{\mathcal{D}}^{(\bullet,0)}$ from the holomorphic super-Clifford algebra onto the holomorphic differential algebra can be most conveniently defined as

$$\begin{gathered}\sigma_{\mathcal{D}}^{(k,0)} : Cl_{\mathcal{D}}^{(k,0)}\mathcal{A} \longrightarrow \Omega_{\mathcal{D}}^{(k,0)}\mathcal{A}, \\ \sigma_{\mathcal{D}}^{(k,0)}(\eta^{\alpha_1}\cdots\eta^{\alpha_k} w_{\alpha_1\cdots\alpha_k}) = z^{\alpha_1}\cdots z^{\alpha_k} w_{\alpha_1\cdots\alpha_k},\end{gathered} \tag{4.10}$$

where (z^{α_i}) denote the basis 1-forms which are complex, Grassmann-even, vectors with two components, i.e. $z \in \mathbb{C}^2$ and

$$z^\alpha z^\beta - z^\beta z^\alpha = 0. \tag{4.11}$$

The holomorphic differential d_h on $\Omega^h_{\mathcal{D}}\mathcal{A}$ is a differential of degree $(1,0)$

$$d_h : \Omega^{(k,0)}_{\mathcal{D}}\mathcal{A} \longrightarrow \Omega^{(k+1,0)}_{\mathcal{D}}\mathcal{A}$$

$$d_h(z^{\alpha_1}\cdots z^{\alpha_k} w_{\alpha_1\cdots\alpha_k}) = \frac{1}{k+1} z^{\alpha_1}\cdots z^{\alpha_{k+1}} \left(\sum_{l=1}^{k+1} D_{\alpha_l} w_{\alpha_1\cdots\alpha_{l-1}\alpha_{l+1}\cdots\alpha_{k+1}}\right). \tag{4.12}$$

Since

$$D_\alpha D_\beta + D_\beta D_\alpha = 0 \tag{4.13}$$

it is

$$d_h^2 = 0. \tag{4.14}$$

Also it follows from this anticommutation relation by the graded Jacobi identity that for any $w_{\alpha_1\cdots\alpha_k} = a_0[D_{\alpha_1}, a_1]\cdots[D_{\alpha_k}, a_k]$ it is

$$d_h(z^{\alpha_1}\cdots z^{\alpha_k} w_{\alpha_1\cdots\alpha_k}) = z^{\alpha_0}\cdots z^{\alpha_k}[D_{\alpha_0}, a_0][D_{\alpha_1}, a_1]\cdots[D_{\alpha_k}, a_k], \tag{4.15}$$

which shows that $(\Omega^h_{\mathcal{D}}\mathcal{A}, d_h)$ describe correctly the pure holomorphic part of $\Omega_{\mathcal{D}}\mathcal{A}$.

The antiholomorphic part of $\Omega_{\mathcal{D}}\mathcal{A}$ can be obtained by analogous arguments or, alternatively, by the fact that the holomorphic and antiholomorphic part are related by hermitean conjugation:

$$(Cl^{(k,0)}_{\mathcal{D}}\mathcal{A})^* = Cl^{(0,k)}_{\mathcal{D}}\mathcal{A} \;\Rightarrow\; (\mathcal{J}^{(k,0)}_{\mathcal{D}})^* = \mathcal{J}^{(0,k)}_{\mathcal{D}} \tag{4.16}$$

and thus one finds for the anti-holomorphic differential algebra $\Omega^{\bar{h}}_{\mathcal{D}}\mathcal{A}$

$$\sigma^{(0,k)}_{\mathcal{D}}: Cl^{(0,k)}_{\mathcal{D}}\mathcal{A} \longrightarrow \Omega^{(0,k)}_{\mathcal{D}}\mathcal{A},$$

$$\sigma^{(0,k)}_{\mathcal{D}}(\overline{\eta}_{\dot\alpha_1}\cdots\overline{\eta}_{\dot\alpha_k} w^{\dot\alpha_1\cdots\dot\alpha_k}) = \overline{z}_{\dot\alpha_1}\cdots\overline{z}_{\dot\alpha_k} w^{\dot\alpha_1\cdots\dot\alpha_k}, \tag{4.17}$$

where $(\overline{z}_{\dot\alpha_i})$ denotes the complex conjugate of (z^{α_i}). The antiholomorphic differential $d_{\bar{h}}$ also is related to d_h by complex conjugation and is given as

$$d_{\bar{h}}: \Omega^{(0,k)}_{\mathcal{D}}\mathcal{A} \longrightarrow \Omega^{(0,k+1)}_{\mathcal{D}}\mathcal{A},$$

$$d_{\bar{h}}(\overline{z}_{\dot\alpha_1}\cdots\overline{z}_{\dot\alpha_k} w^{\dot\alpha_1\cdots\dot\alpha_k}) = \frac{1}{k+1}\overline{z}_{\dot\alpha_1}\cdots\overline{z}_{\dot\alpha_{k+1}}\left(\sum_{l=1}^{k+1}\overline{D}^{\dot\alpha_l} w^{\dot\alpha_1\cdots\dot\alpha_{l-1}\dot\alpha_{l+1}\cdots\dot\alpha_{k+1}}\right). \tag{4.18}$$

Again it is

$$d_{\bar{h}}^2 = 0. \tag{4.19}$$

Having computed the purely holomorphic and antiholomorphic part of $\Omega_{\mathcal{D}}\mathcal{A}$ we now turn to the mixed forms of $\Omega_{\mathcal{D}}\mathcal{A}$, i.e. to $\Omega^{(k,l)}_{\mathcal{D}}\mathcal{A}$ with $k \neq 0$ and $l \neq 0$. Thus we have to determine the correct product of holomorphic and antiholomorphic forms such that the total differential $d_{\mathcal{D}}$ on $\Omega_{\mathcal{D}}\mathcal{A}$ is nilpotent:

$$d_{\mathcal{D}}^2 = 0. \tag{4.20}$$

Since $d_{\mathcal{D}}$ is determined by its action on holomorphic and antiholomorphic forms, it is

$$d_{\mathcal{D}} = d_h + d_{\overline{h}} \tag{4.21}$$

and hence the product of holomorphic and antiholomorphic forms has to be defined such that

$$d_h d_{\overline{h}} + d_{\overline{h}} d_h = 0. \tag{4.22}$$

However, since D_α and $\overline{D}_{\dot\alpha}$ do not anticommute, Eq. (4.22) is not fulfilled for the product which is induced from $Cl_{\mathcal{D}}\mathcal{A}$. Thus there is a nontrivial ideal in $Cl_{\mathcal{D}}\mathcal{A}$ which is generated by $\mathcal{J}_{\mathcal{D}}^{(1,1)}$.

$\mathcal{J}_{\mathcal{D}}^{(1,1)}$ itself is generated by elements of the form $[\overline{D}_{\dot\alpha}, a][D_\alpha, b_+]$ and $[D_\alpha, a'][\overline{D}_{\dot\alpha}, b_-]$ which obey

$$a[D_\alpha, b_+] = 0, \quad a'[\overline{D}_{\dot\alpha}, b_-] = 0, \quad a, a' \in \mathcal{A}, \;\; b_+ \in \mathcal{A}_+, \;\; b_- \in \mathcal{A}_-. \tag{4.23}$$

At this point the S-numbers become important. This can already be seen at first component of a superfield of the form $w = \pi_{\mathcal{D}}(\delta\nu)$, $\nu \in \Omega^1\mathcal{A}$ (we use the same labeling of the components of superfields as in the expansion in θ, $\overline{\theta}$ of Eq. (2.8)). For $\pi_{\mathcal{D}}(\nu) = \eta^\alpha a[D_\alpha, b_+]$ we compute for the first component of w in the $\theta, \overline{\theta}$-expansion

$$\begin{aligned} w_{1\dot\alpha\alpha}|_{S=2} &= a_{3\dot\alpha} b_{2\alpha}|_{S=2} &&= v_{3\alpha\dot\alpha}|_{S=2}, \\ w_{1\dot\alpha\alpha}|_{S=-2} &= a_{3\dot\alpha} b_{2\alpha}|_{S=-2} &&= v_{3\alpha\dot\alpha}|_{S=-2}, \\ w_{1\dot\alpha\alpha}|_{S=0} &= a_{3\dot\alpha} b_{2\alpha}|_{S=0} &&\neq v_{3\alpha\dot\alpha}|_{S=0}, \end{aligned} \tag{4.24}$$

whereas the $S = 0$ part of the third component of v in the $\theta, \overline{\theta}$-expansion is given as

$$v_{3\alpha\dot\alpha}|_{S=0} = a_{3\dot\alpha} b_{2\alpha}|_{S=0} - 2\mathrm{i} a_1 \partial\!\!\!/_{\alpha\dot\alpha} b_1. \tag{4.25}$$

Therefore we conclude that the first component of superfields in $Cl_{\mathcal{D}}^{(1,1)}\mathcal{A}$ with $S = \pm 2$ are never in $\mathcal{J}_{\mathcal{D}}^{(1,1)}$. However, we also see from Eq. (4.25) that there is for any $w_{1\dot\alpha\alpha}|_{S=0}$, given as in Eq. (4.24), an element $\nu \in \Omega^1\mathcal{A}$ such that

$$\pi_{\mathcal{D}}(\nu) = 0 \quad \text{and} \quad \pi_{\mathcal{D}}(\delta\nu)_1 = \eta^\alpha \overline{\eta}_{\dot\alpha} w_{1\alpha}^{\dot\alpha}|_{S=0}. \tag{4.26}$$

Strictly speaking, for $\pi_{\mathcal{D}}(\nu) = 0$ it is not sufficient that $\pi(\nu)_3 = 0$. However, it is straightforward to check that one can arrange a and b_+ such that all other components of $\pi_{\mathcal{D}}(\nu)$ in the $\theta, \overline{\theta}$-expansion vanishes.

Thus we have identified all elements of the form $a_1^{(i)} \partial\!\!\!/_{\alpha\dot\alpha} b_1^{(i)}$ with $a_1^{(i)}, b_1^{(i)} \in C(\mathcal{M})$ as elements of $\mathcal{J}_{\mathcal{D}}^{(1,1)}$. Since $\mathcal{J}$ is an ideal we can multiply such elements with arbitrary elements of $\mathcal{A}$ and obtain

$$\mathcal{N} = \left\{ \sum_i a^{(i)} \partial\!\!\!/_{\alpha\dot\alpha} b^{(i)}, \;\; a^{(i)}, b^{(i)} \in \mathcal{A} \right\} \subseteq \mathcal{J}_{\mathcal{D}}^{(1,1)}. \tag{4.27}$$

Note that if we had not generated the algebra $\mathcal{A}$ by a spinor doublet which are distinguished by the S-numbers, the space $\mathcal{J}_{\mathcal{D}}^{(1,1)}$ would be the whole space $Cl_{\mathcal{D}}^{(1,1)}\mathcal{A}$ and thus there would be no differential form with both holomorphic and antiholomorphic indices. The reason for this is that due to the S-numbers the range of the ∂_α-part of D_α is larger than the range of the $\mathrm{i}\,\partial\!\!\!/_{\alpha\dot\alpha}\overline{\theta}^{\dot\alpha}$-part of D_α which would not be the case if there is no split of components caused by S-numbers.

What remains to be shown is that we have determined all of $\mathcal{J}_{\mathcal{D}}^{(1,1)}$, i.e. we can replace the "$\subseteq$" by "$=$" in Eq. (4.27). For this purpose it is convenient to define a projection-operator $P_S^{(n)}$ which projects the components of the $\theta,\overline{\theta}$-expansion of any superfield $w \in Cl_{\mathcal{D}}^{(n-k,k)}\mathcal{A}$ onto the parts with the highest S-numbers i.e. for $w \in Cl_{\mathcal{D}}^{(k,l)}\mathcal{A}$ with $k+l=n$, $k\neq 0$, $l\neq 0$ and

$$\begin{aligned}
&|S(w_1)| = n, \qquad |S(w_2)| = |S(w_3)| = n+1,\\
&|S(w_4)| = |S(w_5)| = |S(w_6)| = n+2, \\
&|S(w_7)| = |S(w_8)| = n+3, \qquad |S(w_9)| = n+4,
\end{aligned} \tag{4.28}$$

it is

$$P_S^{(n)}(w) = w \tag{4.29}$$

and $P_S^{(n)}(w) = 0$ for all $w \in Cl_{\mathcal{D}}^{(k,l)}\mathcal{A}$ which do not have components with S-numbers as in Eq. (4.28). For later convenience we extend the definition of P_S to the holomorphic and antiholomorphic part of $Cl_{\mathcal{D}}\mathcal{A}$:

$$P_S^{(n)} w = w \quad \forall w \in (Cl_{\mathcal{D}}^{(n,0)}\mathcal{A} + Cl_{\mathcal{D}}^{(0,n)}\mathcal{A}). \tag{4.30}$$

Before we discuss the ideal generated by $\mathcal{N}$ let us check that at the level of 2-forms $\mathcal{N}$ is the correct space by which one has to divide $Cl_{\mathcal{D}}^{(1,1)}\mathcal{A}$ in order to obtain a well-defined differential. First we note that

$$\ker P_S^{(2)} \cap Cl_{\mathcal{D}}^{(1,1)}\mathcal{A} = \mathcal{N}. \tag{4.31}$$

Let $\pi_{\mathcal{D}}(a^{(i)}\delta b^{(i)}) = v$ be an arbitrary 1-form. We compute for the components of $P_S^{(2)}$ $(\pi_{\mathcal{D}}(\delta a^{(i)}\delta b^{(i)}))$ with a holomorphic and an antiholomorphic index

$$\begin{aligned}
&P_S^{(2)}([D_\alpha, a^{(i)}][\overline{D}_{\dot\alpha}, b^{(i)}]) + P_S^{(2)}([\overline{D}_{\dot\alpha}, a^{(i)}][D_\alpha, b^{(i)}])\\
&\quad = P_S^{(2)}([\partial_\alpha, a^{(i)}][\overline{\partial}_{\dot\alpha}, b^{(i)}]) + P_S^{(2)}([\overline{\partial}_{\dot\alpha}, a^{(i)}][\partial_\alpha, b^{(i)}])\\
&\quad = P_S^{(2)}([\partial_\alpha, a^{(i)}[\overline{\partial}_{\dot\alpha}, b^{(i)}]]_+) + P_S^{(2)}([\overline{\partial}_{\dot\alpha}, a^{(i)}[\partial_\alpha, b^{(i)}]]_+)\\
&\quad = P_S^{(2)}([\partial_\alpha, v]_+) + P_S^{(2)}([\overline{\partial}_{\dot\alpha}, v]_+).
\end{aligned} \tag{4.32}$$

From this we conclude that if $v=0$ then it is

$$P_S^{(2)}([D_\alpha, a^{(i)}][\overline{D}_{\dot\alpha}, b^{(i)}]) + P_S^{(2)}([\overline{D}_{\dot\alpha}, a^{(i)}][D_\alpha, b^{(i)}]) = 0 \tag{4.33}$$

which implies that

$$\mathcal{N} = \mathcal{J}_{\mathcal{D}}^{(1,1)}. \tag{4.34}$$

We now turn to the ideal which is generated by $\mathcal{N}$. With the definition of P_S given in Eqs. (4.28) and (4.30) it is straightforward to check that it is

$$P_S^{(k+l+m+n)}(w_1 w_2) = P_S^{(k+l)}(w_1) P_S^{(m+n)}(w_2), \quad w_1 \in Cl_{\mathcal{D}}^{(k,l)}\mathcal{A},\ w_2 \in Cl_{\mathcal{D}}^{(m,n)}\mathcal{A} \tag{4.35}$$

and

$$P_S^{(k+l+2)}(w_1 w_2) = 0, \quad w_1 \in Cl_{\mathcal{D}}^{(k,l)}\mathcal{A},\ w_2 \in \mathcal{N}. \tag{4.36}$$

Thus the ideal I generated by $\mathcal{N}$ is given by the kernels of the projectors $P_S^{(n)}$, $n \in \mathbb{N}$:

$$I = \bigcup_{n \geq 2} \ker P_S^{(n)}. \tag{4.37}$$

From this it follows that product of holomorphic and antiholomorphic forms is defined as follows: for any $v = z^{\alpha_1} \cdots z^{\alpha_k} v_{\alpha_1 \cdots \alpha_k} \in \Omega_{\mathcal{D}}^{(k,0)}\mathcal{A}$ and $w = \bar{z}_{\dot{\alpha}_1} \cdots \bar{z}_{\dot{\alpha}_l} w^{\dot{\alpha}_1 \cdots \dot{\alpha}_l} \in \Omega_{\mathcal{D}}^{(0,l)}\mathcal{A}$ it is

$$vw = z^{\alpha_1} \cdots z^{\alpha_k} \bar{z}_{\dot{\alpha}_1} \cdots \bar{z}_{\dot{\alpha}_l} P_S^{(k+l)}(v_{\alpha_1 \cdots \alpha_k} w^{\dot{\alpha}_1 \cdots \dot{\alpha}_l}). \tag{4.38}$$

Thus we define the map $\sigma_{\mathcal{D}}^{(\bullet,\bullet)}$ from the super-Clifford algebra to the superdifferential algebra for any $k, l \in \mathbb{N}$:

$$\begin{aligned} \sigma_{\mathcal{D}}^{(k,l)} &: Cl_{\mathcal{D}}^{(k,l)}\mathcal{A} \longrightarrow \Omega_{\mathcal{D}}^{(k,l)}\mathcal{A} \\ \sigma_{\mathcal{D}}^{(k,l)}\left(\eta^{\alpha_1} \cdots \eta^{\alpha_k} \bar{\eta}_{\dot{\alpha}_1} \cdots \bar{\eta}_{\dot{\alpha}_l} w_{\alpha_1 \cdots \alpha_k}^{\dot{\alpha}_1 \cdots \dot{\alpha}_l}\right) &= z^{\alpha_1} \cdots z^{\alpha_k} \bar{z}_{\dot{\alpha}_1} \cdots \bar{z}_{\dot{\alpha}_l} P_S^{(k+l)}\left(w_{\alpha_1 \cdots \alpha_k}^{\dot{\alpha}_1 \cdots \dot{\alpha}_l}\right). \end{aligned} \tag{4.39}$$

The extension of d_h and $d_{\bar{h}}$ to the mixed forms in $\Omega_{\mathcal{D}}^{(k,l)}\mathcal{A}$ is obtained with the help of the projection P_S:

$$\begin{aligned} d_h &: \Omega_{\mathcal{D}}^{(k,l)}\mathcal{A} \longrightarrow \Omega_{\mathcal{D}}^{(k+1,l)}\mathcal{A} \\ d_h(w) &= z^\alpha P_S^{(k+l+1)}(D_\alpha w - (-1)^{(k+l)} w D_\alpha) = z^\alpha P_S^{(k+l+1)}([D_\alpha, w]) \end{aligned} \tag{4.40}$$

and

$$\begin{aligned} d_{\bar{h}} &: \Omega_{\mathcal{D}}^{(k,l)}\mathcal{A} \longrightarrow \Omega_{\mathcal{D}}^{(k,l+1)}\mathcal{A} \\ d_{\bar{h}}(w) &= z_{\dot{\alpha}} P_S^{(k+l+1)}(\bar{D}^{\dot{\alpha}} w - (-1)^{(k+l)} w \bar{D}^{\dot{\alpha}}) = z_{\dot{\alpha}} P_S^{(k+l+1)}([\bar{D}^{\dot{\alpha}}, w]). \end{aligned} \tag{4.41}$$

The nilpotency of the holomorphic and antiholomorphic differential is again ensured by the symmetrization of the holomorphic and antiholomorphic indices. What remains to be checked is that

$$d_h d_{\bar{h}} + d_{\bar{h}} d_h = 0. \tag{4.42}$$

However, this equation can be verified by the following computation: For any $w \in \Omega_{\mathcal{D}}^{(k,l)}\mathcal{A}$ it is

$$d_h(d_{\overline{h}}w) = \overline{z}_{\dot\alpha} d_h(P_S^{(k+l+1)}(\overline{D}^{\dot\alpha} w)) = z^\alpha \overline{z}_{\dot\alpha} P_S^{(k+l+2)}(D_\alpha(P_S^{(k+l+1)}(\overline{D}^{\dot\alpha} w)))$$
$$= z^\alpha \overline{z}_{\dot\alpha} P_S^{(k+l+2)}(D_\alpha \overline{D}^{\dot\alpha} w) \tag{4.43}$$

and also

$$d_{\overline{h}}(d_h w) = z^\alpha d_{\overline{h}}(P_S^{(k+l+1)}(D_\alpha w)) = z^\alpha \overline{z}_{\dot\alpha} P_S^{(k+l+2)}(\overline{D}^{\dot\alpha}(P_S^{(k+l+1)}(D_\alpha w)))$$
$$= z^\alpha \overline{z}_{\dot\alpha} P_S^{(k+l+2)}(\overline{D}^{\dot\alpha} D_\alpha w). \tag{4.44}$$

Thus it is

$$(d_h d_{\overline{h}} + d_{\overline{h}} d_h)(w) = z^\alpha \overline{z}_{\dot\alpha} P_S^{(k+l+2)}((D_\alpha \overline{D}^{\dot\alpha} + \overline{D}^{\dot\alpha} D_\alpha) w)$$
$$= z^\alpha \overline{z}^{\dot\alpha} P_S^{(k+l+2)}(2\mathrm{i}\,\partial\!\!\!/_{\alpha\dot\alpha} w) = 0. \tag{4.45}$$

From this it follows that $d_{\mathcal{D}} = d_h + d_{\overline{h}}$ is a nilpotent differential with $d_{\mathcal{D}}^2 = 0$ on n-forms $n \in \mathbb{N}$. This completes the construction of $\Omega_{\mathcal{D}}\mathcal{A}$.

Note that the generalized differential algebra $\Omega_{\mathcal{D}}\mathcal{A}$ itself does not contain any information of the underlying manifold $\mathcal{M}$ in the sense that the differential forms in $\Omega_{\mathcal{D}}\mathcal{A}$ and the differential do not depend on space–time derivatives. Although for the construction of $\Omega_{\mathcal{D}}\mathcal{A}$ the presence of $\mathcal{M}$ played an important role, it turned out that all dependence on $C^\infty(\mathcal{M})$, via the $\partial\!\!\!/$-part of $\mathcal{D}$, is contained in the differential ideal $\mathcal{J}$. Thus $\Omega_{\mathcal{D}}\mathcal{A}$ is a generalized differential algebra associated to the finite-dimensional Grassmann algebra in $\mathcal{A}$ which is multiplied by $C^\infty(\mathcal{M})$. For differential forms in $\Omega_{\mathcal{D}}\mathcal{A}$ which have both, holomorphic and antiholomorphic indices, this statement is a direct consequence of Eq. (4.32).

For the pure holomorphic forms one can perform a change of coordinates

$$x^\mu \longrightarrow y_-^\mu = x^\mu - \mathrm{i}\theta\sigma^\mu\overline{\theta}. \tag{4.46}$$

This induces the following transformation of the operators D_α and $\overline{D}_{\dot\alpha}$:

$$D_\alpha \longrightarrow D_\alpha^{(-)} = \partial_\alpha, \qquad \overline{D}_{\dot\alpha} \longrightarrow \overline{D}_{\dot\alpha}^{(-)} = -\overline{\partial}_{\dot\alpha} - 2\mathrm{i}\theta^\alpha \partial\!\!\!/_{\alpha\dot\alpha}. \tag{4.47}$$

Since the pure holomorphic forms are built only out of commutators with D it follows from Eq. (4.47) that they do not depend on space–time derivatives.

For the pure antiholomorphic forms there is a similar change of coordinates

$$x^\mu \longrightarrow y_+^\mu = x^\mu + \mathrm{i}\theta\sigma^\mu\overline{\theta}, \tag{4.48}$$

which leads to

$$D_\alpha \longrightarrow D_\alpha^{(+)} = \partial_\alpha + 2\mathrm{i}\,\partial\!\!\!/_{\alpha\dot\alpha}\overline{\theta}, \qquad \overline{D}_{\dot\alpha} \longrightarrow \overline{D}_{\dot\alpha}^{(+)} = -\overline{\partial}_{\dot\alpha}. \tag{4.49}$$

Thus we conclude that the pure antiholomorphic forms do not depend on space–time derivatives.

Furthermore, we observe that differential forms with holomorphic and antiholomorphic indices are invariant under transformations (4.46) and (4.48), i.e.

$$P_S^{(k+l)}(\omega(x)) = P_S^{(k+l)}(\omega(y_+)) = P_S^{(k+l)}(\omega(y_-)) \quad \forall \omega \in Cl_{\mathcal{D}}^{(k,l)}\mathcal{A},\ k,l > 0, \tag{4.50}$$

which is a direct consequence of Eqs. (4.47) and (4.49).

As a result of this discussion we may relate the generalized differential algebra $\Omega_{\mathcal{D}}\mathcal{A}$ to the algebra $C^\infty(\mathcal{M}) \otimes \Lambda((\theta, \overline{\theta}) \times \mathbb{C}^2)$, where $\Lambda((\theta, \overline{\theta}) \times \mathbb{C}^2)$ denotes the $\mathbb{Z}_2$-graded analog of the de Rham algebra over the Grassmann algebra generated by $(\theta, \overline{\theta}) \times \mathbb{C}^2$. Such $\mathbb{Z}_2$-graded de Rham algebras have already been studied in the framework of noncommutative geometry in [21] where the relation between closed de Rham currents and cyclic cocycles over a Grassmann algebra was established.

However, the algebra $\Omega_{\mathcal{D}}\mathcal{A}$ is not isomorphic to $C^\infty \otimes \Lambda((\theta, \overline{\theta}) \times \mathbb{C}^2)$ because of the projection operator P_S. The definition of P_S in Eqs. (4.28),(4.29) and (4.30) can naturally be transferred to $\Lambda((\theta, \overline{\theta}) \times \mathbb{C}^2)$. With this projection operator P_S defined on $C^\infty \otimes \Lambda((\theta, \overline{\theta}) \times \mathbb{C}^2)$ (where P_S is extended by the identity on $C^\infty(\mathcal{M})$) it is

$$\Omega_{\mathcal{D}}\mathcal{A} = P_S(C^\infty(\mathcal{M}) \otimes \Lambda((\theta, \overline{\theta}) \times \mathbb{C}^2). \tag{4.51}$$

Strictly speaking, the identification of pure holomorphic form and pure antiholomorphic form involves also coordinate transformations of the form Eqs. (4.46) and (4.48).

5. The inner product and supersymmetry transformations

With the generalized differential algebra $\Omega_{\mathcal{D}}\mathcal{A}$ we have all necessary objects at hand to construct the covariant derivative and curvature, the main objects in Yang–Mills theory. However, what is still missing is an inner product on $\Omega_{\mathcal{D}}\mathcal{A}$ which would allow us to define an action. The standard procedure in noncommutative geometry uses the fact that there is a natural inner product on $Cl_{\mathcal{D}}\mathcal{A}$ which induces an inner product on $\Omega_{\mathcal{D}}\mathcal{A}$ [1]. In principle we shall also follow this construction although there will be some important deviations from the usual procedure.

Let us first define an inner product on $Cl_{\mathcal{D}}\mathcal{A}$. Therefore we recall that a general element $\omega \in Cl_{\mathcal{D}}^{(k,l)}\mathcal{A}$ is of the form

$$\eta^{\alpha_1} \cdots \eta^{\alpha_k} \overline{\eta}_{\dot{\alpha}_1} \cdots \overline{\eta}_{\dot{\alpha}_l} \otimes \omega^{\dot{\alpha}_1 \cdots \dot{\alpha}_l}_{\alpha_1 \cdots \alpha_k}, \tag{5.1}$$

where the first factor acts on $\mathcal{H}_{\mathrm{H}}$ and the second factor acts on $\mathcal{H}_\pi$. Using this notation, we define

$$\mathcal{H}_{(k,l)} = \{\omega^{\dot{\alpha}_1 \cdots \dot{\alpha}_l}_{\alpha_1 \cdots \alpha_k} |0\rangle \,|\, \omega \in Cl_{\mathcal{D}}^{(k,l)}\mathcal{A}\}, \tag{5.2}$$

which is completely analogous to the definition of $\mathcal{H}_{\mathcal{A}}$ in Eq. (2.23). Again this allows us to use the inner product on $\mathcal{H}_\pi$ for the definition of a (degenerate) inner product on $Cl_{\mathcal{D}}^{(k,l)}\mathcal{A}$ for any $k, l \geq 0$

$$\langle \omega | \nu \rangle = ((\omega^{\alpha_1 \cdots \alpha_k}_{\dot{\alpha}_1 \cdots \dot{\alpha}_l} |0\rangle), (\nu^{\dot{\alpha}_1 \cdots \dot{\alpha}_l}_{\alpha_1 \cdots \alpha_k} |0\rangle)), \quad \omega, \nu \in Cl_{\mathcal{D}}^{(k,l)}\mathcal{A}. \tag{5.3}$$

This inner product is degenerate for the same reason as the inner product (2.24) on $\mathcal{A}$ is degenerate. However, on the subspaces $(Cl_{\mathcal{D}}^{(k,l)}\mathcal{A})_{v^{k+l-2m}}$, $2m \leq k + l$, the inner product is nondegenerate if one of the components of $v \in \mathbb{C}^2$ is nonzero. The natural inner product on $\mathcal{A}_0^{(k,l)}$ and the inner product defined in Eq. (5.3) are related by

$$\begin{aligned}
&\langle i_{\upsilon^{k+l-2m}}(\omega_0), i_{\upsilon^{k+l-2n}}(\nu_0)\rangle, \\
&= \frac{(2(k+l+2-m-n))!}{((k+l+2-m-n)!)^2}(\upsilon_+\upsilon_-)^{k+l+2-m-n}(\omega_0, \nu_0) \\
&= \frac{(2(k+l+2-m-n))!}{((k+l+2-m-n)!)^2}(\upsilon_+\upsilon_-)^{k+l+2-m-n}\int\limits_{\mathcal{M}}(\omega_0^*\nu_0)|_{\theta\theta\bar\theta\bar\theta}
\end{aligned} \tag{5.4}$$

for all $\omega_0 \in (Cl_{\mathcal{D}}^{(k,l)}\mathcal{A})_{\upsilon^{k+l-2m}}$ and $\nu_0 \in (Cl_{\mathcal{D}}^{(k,l)}\mathcal{A})_{\upsilon^{k+l-2n}}$. Since it is our aim to construct an action which is invariant under supersymmetry transformation we are interested only in inner products on the spaces $(Cl_{\mathcal{D}}^{(k,l)}\mathcal{A})_{\upsilon^{k+l-2m}}$. For later convenience we rescale these inner products

$$\begin{aligned}
&\langle\omega, \nu\rangle_i = \frac{((k+l+2-m-n)!)^2}{(2(k+l+2-m-n))!}(\upsilon_+\upsilon_-)^{-k-l+m+n}\langle\omega, \nu\rangle, \\
&\omega \in (Cl_{\mathcal{D}}^{(k,l)}\mathcal{A})_{\upsilon^{k+l-2m}}, \quad \nu \in (Cl_{\mathcal{D}}^{(k,l)}\mathcal{A})_{\upsilon^{k+l-2n}}
\end{aligned} \tag{5.5}$$

such that $i_{\upsilon^{k+l-2m}}$ becomes an isometric map.

We now come to the discussion about the relation of $\Omega_{\mathcal{D}}\mathcal{A}$ and supersymmetry transformations. Clearly we can transfer the embedding i_υ of tensor superfield from $Cl_{\mathcal{D}}\mathcal{A}$ to $\Omega_{\mathcal{D}}\mathcal{A}$, i.e. we define the subalgebra $(\Omega_{\mathcal{D}}\mathcal{A})_\upsilon \subset \Omega_{\mathcal{D}}\mathcal{A}$ which carries a representation of the supersymmetry algebra for any $k, l \in \mathbb{N}$ as

$$(\Omega_{\mathcal{D}}^{(k,l)}\mathcal{A})_\upsilon = \sigma_{\mathcal{D}}^{(k,l)} \circ i_\upsilon(\mathcal{A}_0^{(k,l)}). \tag{5.6}$$

Note that $\sigma_{\mathcal{D}}^{(k,l)}$ is an invertible homomorphism from $(Cl_{\mathcal{D}}^{(k,l)}\mathcal{A})_{\upsilon^{k+l}}$ to $(\Omega_{\mathcal{D}}^{(k,l)}\mathcal{A})_\upsilon$ if $k, l > 0$. Therefore we can define for any $k, l > 0$

$$c_\upsilon^{(k,l)} : (\Omega_{\mathcal{D}}^{(k,l)}\mathcal{A})_\upsilon \longrightarrow (Cl_{\mathcal{D}}^{(k,l)}\mathcal{A})_{\upsilon^{k+l}} \tag{5.7}$$

as the inverse of $i_{\upsilon^{k+l}}$:

$$c_\upsilon^{(k,l)} = \sigma_{\mathcal{D}}^{(k,l)^{-1}}|_{(Cl_{\mathcal{D}}^{(k,l)}\mathcal{A})_{\upsilon^{k+l}}}. \tag{5.8}$$

This map can be used to define an inner product on $(\Omega_{\mathcal{D}}\mathcal{A})_\upsilon$ which is induced by the inner product on $(Cl_{\mathcal{D}}\mathcal{A})_\upsilon$. However, the invariance under supersymmetry transformations of this product is not automatically guaranteed.

For the pure holomorphic part of $\Omega_{\mathcal{D}}\mathcal{A}$ we find that the image of d_h acting on $(\Omega_{\mathcal{D}}\mathcal{A})_\upsilon$ is not contained in $(\Omega_{\mathcal{D}}\mathcal{A})_\upsilon$

$$d_h(\Omega_{\mathcal{D}}^{(k,0)}\mathcal{A})_\upsilon \not\subset (\Omega_{\mathcal{D}}^{(\cdot,0)}\mathcal{A})_\upsilon. \tag{5.9}$$

The reason for this is the same as the one discussed at the end of Section 3: the $\mathrm{i}\,\not\partial_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}$-part of d_h generates terms which are not in $(\Omega_{\mathcal{D}}\mathcal{A})_\upsilon$. The same is true for the pure antiholomorphic forms and the differential $d_{\bar h}$.

The situation is different for forms with mixed indices since here the disturbing part of the derivative is projected out. Thus it is for all $k, l \in \mathbb{N}$ with $l > 0$

$$d_h\omega \in (\Omega_{\mathcal{D}}^{(k+1,l)}\mathcal{A})_\upsilon, \quad \forall\omega \in (\Omega_{\mathcal{D}}^{(k,l)}\mathcal{A})_\upsilon, \tag{5.10}$$

and for all $k, l \in \mathbb{N}$ with $k > 0$,

$$d_{\bar{h}}\omega \in (\Omega_{\mathcal{D}}^{(k,l+1)}\mathcal{A})_v, \quad \forall \omega \in (\Omega_{\mathcal{D}}^{(k,l)}\mathcal{A})_v. \tag{5.11}$$

However, supersymmetry transformations do not commute with the differentials:

$$\begin{aligned} [d_h, (\bar{\varepsilon}\bar{Q})_v]\omega &= \mathrm{i} z^{\alpha} \not{\partial}_{\alpha\dot{\alpha}} \bar{\varepsilon}^{\dot{\alpha}}_{(v)}\omega, \quad \omega \in (\Omega_{\mathcal{D}}^{(k,l)}\mathcal{A})_v,\ l > 0, \\ [d_{\bar{h}}, (\varepsilon Q)_v]\omega &= \mathrm{i}\bar{z}_{\dot{\alpha}} \bar{\not{\partial}}^{\dot{\alpha}\alpha} \varepsilon_{\alpha(v)}\omega \quad \forall \omega \in (\Omega_{\mathcal{D}}^{(k,l)}\mathcal{A})_v,\ k > 0, \end{aligned} \tag{5.12}$$

where it is $(\varepsilon_{\alpha(v)}, \bar{\varepsilon}^{\dot{\alpha}}{}_{(v)}) = (\varepsilon_{\alpha}, \bar{\varepsilon}^{\dot{\alpha}}) \otimes v$.

On the other hand, it is for any $k, l > 0$ and $0 < 2m \leq k + l$

$$(Cl_{\mathcal{D}}^{(k,l)}\mathcal{A})_{v^{k+l-2m}} \subset \mathcal{J}_{\mathcal{D}}^{(k,l)}. \tag{5.13}$$

Thus it is for any $\omega \in (\Omega_{\mathcal{D}}^{(k,l)}\mathcal{A})_v$

$$d_h\omega = z^{\alpha} P_S^{(k+l+1)}(\partial_{\alpha} c_v^{(k,l)}(\omega)) = z^{\alpha} P_S^{(k+l+1)}(D_{\alpha} c_v^{(k,l)}(\omega)). \tag{5.14}$$

Although $\omega' = \eta^{\alpha} P_S^{(k+l+1)}(D_{\alpha} c_v^{(k,l)}(\omega))$ also is not covariant under supersymmetry transformations, the product of ω' with any other element in $(\Omega_{\mathcal{D}}^{(k,l)}\mathcal{A})_v$ with covariant transformation properties under supersymmetry transformations is invariant, i.e. for any $\nu(\Omega_{\mathcal{D}}^{(k+1,l)}\mathcal{A})_v$ with $\nu = \sigma_{\mathcal{D}}^{(k+1,l)} \circ i_{v^{k+l+1}}(\nu_0)$ and $\omega = \sigma_{\mathcal{D}}^{(k,l)} \circ i_{v^{k+l}}(\omega_0)$ it is

$$\begin{aligned} \langle c_v^{(k+1,l)}(\nu), \omega' \rangle_i &= \langle c_v^{(k+1,l)}, \partial_{\alpha} c_v^{(k,l)}(\omega) + \mathrm{i} \not{\partial}_{i\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} c_v^{(k,l)}(\omega) \rangle_i \\ &= \langle c_v^{(k+1,l)}(\nu), i_{v^{k+l+1}}(\partial_{\alpha}\omega_0) + i_{v^{k+l-1}}(\mathrm{i}\not{\partial}_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} \omega_0) \rangle_i \\ &= \int_{\mathcal{M}} (\nu_0^* D_{\alpha}\omega_0)|_{\theta\theta\bar{\theta}\bar{\theta}}. \end{aligned} \tag{5.15}$$

The same arguments apply for the antiholomorphic derivative, i.e. for $d_h\omega \in (\Omega_{\mathcal{D}}^{(k,l+1)}\mathcal{A})_v$ the product $\langle c_v^{(k,l+1)}(\nu), \bar{D}^{\dot{\alpha}} c_v^{(k,l)}\omega \rangle_i$ is invariant under supersymmetry transformations for all $\nu \in (\Omega_{\mathcal{D}}^{(k,l+1)}\mathcal{A})_v$.

6. Supersymmetric Yang–Mills theory

Once the generalized differential algebra $\Omega_{\mathcal{D}}\mathcal{A}$ is known the covariant derivative and curvature can be defined [1]. We repeat from this general procedure only the basic definitions which allows us to fix our notation. A comprehensive presentation of this topic can be found in [1,19,20].

The covariant derivative is defined with respect to some gauge group which is in this framework

$$\mathbf{G} = \{u \in \mathcal{A} | uu^* = u^*u = 1\}. \tag{6.1}$$

There is a representation of this group on $\mathcal{H}$, the Hilbert space of the spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ which is given by π, the representation of $\mathcal{A}$. The operator $\mathcal{D}$ can be extended to a covariant derivative by adding a connection 1-form $A \in \Omega^1_{\mathcal{D}}\mathcal{A} \cong \pi_{\mathcal{D}}(\Omega^1\mathcal{A})$, i.e. we define the covariant derivative as an operator acting on $\mathcal{H}$ by

$$\nabla\!\!\!/ = \mathcal{D} + A\!\!\!/, \tag{6.2}$$

where $A\!\!\!/ \in \pi_{\mathcal{D}}(\Omega^1\mathcal{A})$ is hermitean and obeys the following transformation rule

$$A\!\!\!/ \longrightarrow A\!\!\!/' = u A\!\!\!/ u^* + u\mathcal{D}u^*. \tag{6.3}$$

The operator $\nabla\!\!\!/$ transforms covariant under gauge transformations

$$\nabla\!\!\!/ \longrightarrow \nabla\!\!\!/' = u \nabla\!\!\!/ u^*. \tag{6.4}$$

Alternatively, the covariant derivative can be defined as an operator acting on forms, i.e. as an operator acting on $\Omega_{\mathcal{D}}\mathcal{A} \otimes \mathcal{H}$

$$\nabla = d_{\mathcal{D}} + A, \tag{6.5}$$

where $A = \sigma^1_{\mathcal{D}}(A\!\!\!/) \in \Omega^1_{\mathcal{D}}\mathcal{A}$ denotes the 1-form corresponding to $A\!\!\!/$. Of course, ∇ also transforms covariantly under gauge transformations.

The curvature F is defined as the square of the covariant derivative

$$F = \nabla\nabla = d_{\mathcal{D}}A + AA \tag{6.6}$$

and it is easy to show that also in the general framework of noncommutative geometry this definition leads to a 2-form, i.e. $F \in \Omega^2_{\mathcal{D}}\mathcal{A}$, which transforms covariantly under gauge transformations.

Let us now apply this general construction to the case where $\mathcal{A}_n$ is the tensor product of the commutative algebra of superfields as defined in Section 4 and the algebra of complex $n \times n$-matrices, $M_{n\times n}(\mathbb{C})$, i.e.

$$\mathcal{A}_n = \mathcal{A} \otimes M_{n\times n}(\mathbb{C}). \tag{6.7}$$

The representation space $\mathcal{H}$ has to be extended by a representation of $M_{n\times n}$ such that it becomes a representation space $\mathcal{H}_n$ of $\mathcal{A}_n$. The only irreducible representation of the associative algebra $M_{n\times n}(\mathbb{C})$ is $\mathbb{C}^n$. Thus we take this irreducible representation and obtain for $\mathcal{H}_n$

$$\mathcal{H}_n = \mathcal{H} \otimes \mathbb{C}^n. \tag{6.8}$$

The operator $\mathcal{D}$ is extended trivially to an operator $\mathcal{D}_n$

$$\mathcal{D}_n = \mathcal{D} \otimes 1_{n\times n}, \tag{6.9}$$

where $\mathcal{D}$ is defined as in Eq. (2.34). As a consequence of this setting the generalized differential forms in $\Omega_{\mathcal{D}}\mathcal{A}$ become matrix-valued generalized differential-forms.

The gauge group $\mathbf{G}$ is the group of superfields which are generated by the super-Lie algebra $\mathbf{g}$:

$$\mathbf{g} = \{\Lambda \in \mathcal{A} |\ \Lambda^* = \Lambda\}. \tag{6.10}$$

Thus any $u \in \mathbf{G}$ can be written as $u = \exp(\mathrm{i}\Lambda)$, $\Lambda \in \mathbf{g}$. Obviously the first component of any $u \in \mathbf{G}$ of the θ, $\overline{\theta}$-expansion is a bosonic $U(n)$-gauge transformation. However, any $u \in \mathbf{G}$ represents a full superfield and therefore the bosonic gauge group is extended by a nilpotent part, containing also Grassmann-odd transformations.

We saw that the derivative $d_{\mathcal{D}}$ of the $\Omega_{\mathcal{D}}\mathcal{A}$, constructed in the previous sections, splits into a holomorphic part d_h and an antiholomorphic part $d_{\overline{h}}$. Also the space of 1-forms $\Omega^1_{\mathcal{D}}\mathcal{A}$ can be decomposed into a holomorphic part $\Omega^{(1,0)}_{\mathcal{D}}\mathcal{A}$ and an antiholomorphic part $\Omega^{(0,1)}_{\mathcal{D}}\mathcal{A}$. Thus we can introduce the holomorphic and antiholomorphic derivative

$$\nabla = \nabla_h + \nabla_{\overline{h}}, \quad \nabla_h = d_h + A_h, \quad \nabla_{\overline{h}} = d_{\overline{h}} + A_{\overline{h}}, \tag{6.11}$$

where $A_h = z^\alpha A_\alpha$, resp. $A_{\overline{h}} = \overline{z}_{\dot\alpha} A^{\dot\alpha}$ denotes the holomorphic, resp. antiholomorphic part of $A = A_h + A_{\overline{h}}$.

This split propagates to the 2-forms where we can decompose the curvature as follows:

$$F = F_h + F_{\overline{h}} + F_v \tag{6.12}$$

with

$$F_h = \nabla_h^2 = d_h A_h + A_h A_h, \qquad F_{\overline{h}} = \nabla_{\overline{h}}^2 = d_{\overline{h}} A_{\overline{h}} + A_{\overline{h}} A_{\overline{h}} \tag{6.13}$$

and

$$F_v = \nabla_h \nabla_{\overline{h}} + \nabla_{\overline{h}} \nabla_h = d_h A_{\overline{h}} + d_{\overline{h}} A_h + A_h A_{\overline{h}} + A_{\overline{h}} A_h. \tag{6.14}$$

As in the usual approach to supersymmetric gauge theory the full curvature contains superfluous components [14] and one has to get rid of them without spoiling covariance. The standard procedure is to impose the constraint that all components of F with two spinorial indices vanish. In our case, this clearly would be too strong since it would imply that the complete curvature vanishes. However, the standard constraints in the usual approach have different reasonings: the requirement that the vectorial part of the curvature, i.e. $F_{\alpha\dot\alpha}$, should vanish is simply a redefinition of fields which is possible because of the presence of the torsion term. This torsion term is absent in our approach. Therefore the constraint $F_{\alpha\dot\alpha} = 0$ would be a real restriction and thus we drop this constraint.

The other constraints arise as a consequence of the chirality conditions which reads

$$\nabla_h \overline{\Phi} = 0, \quad \nabla_{\overline{h}} \Phi = 0, \qquad \Phi, \overline{\Phi} \in \mathcal{H}_\pi. \tag{6.15}$$

These conditions can be applied consistently only if

$$\nabla_h \nabla_h = F_h = 0, \qquad \nabla_{\overline{h}} \nabla_{\overline{h}} = F_{\overline{h}} = 0. \tag{6.16}$$

This leads to the same restrictions on A as in the conventional approach. In components the constraints read

$$\begin{aligned} F_{\alpha\beta} &= D_\alpha A_\beta + D_\beta A_\alpha + A_\alpha A_\beta + A_\beta A_\beta = 0 \\ F_{\dot\alpha\dot\beta} &= \overline{D}_{\dot\alpha} A_{\dot\beta} + \overline{D}_{\dot\beta} A_{\dot\alpha} + A_{\dot\alpha} A_{\dot\beta} + A_{\dot\beta} A_{\dot\alpha} = 0. \end{aligned} \tag{6.17}$$

The most general solution to the constraints in Eq. (6.17) are

$$A_\alpha = T^{-1} D_\alpha T, \quad A_{\dot\alpha} = S^{-1} \overline{D}_{\dot\alpha} S, \tag{6.18}$$

where $T, S \in \mathcal{A}$ are general invertible superfields. They are related by the requirement that $\nabla\!\!\!/$ is a self-adjoint operator. Thus it is $A\!\!\!/^* = A\!\!\!/$ and hence $A_{h\alpha} = -A_{\overline{h}\dot\alpha}$. This implies

$$S^* = T^{-1}. \tag{6.19}$$

Inserting this result in Eq. (6.14) we obtain for the remaining part of the curvature:

$$\begin{aligned} F_{\alpha\dot\alpha} = P_S^2(&D_\alpha(T^*\overline{D}_{\dot\alpha}(T^{-1})^*) + \overline{D}_{\dot\alpha}(T^{-1}D_\alpha T) \\ &+(T^*\overline{D}_{\dot\alpha}(T^{-1})^*)(T^{-1}D_\alpha T) + (T^{-1}D_\alpha T)(T^*\overline{D}_{\dot\alpha}(T^{-1})^*)), \end{aligned} \tag{6.20}$$

which can be rewritten as

$$\begin{aligned} F_{\alpha\dot\alpha} &= P_S^{(2)}(T^*\overline{D}_{\dot\alpha}(W^{-1}D_\alpha W)(T^{-1})^* - 2\mathrm{i}T^*\partial\!\!\!/_{\alpha\dot\alpha}(T^{-1})^*) \\ &= P_S^{(2)}(\overline{\partial}_{\dot\alpha}(W^{-1}\partial_\alpha W)) = T^* W_{\alpha\dot\alpha}(T^{-1})^*, \end{aligned} \tag{6.21}$$

where we have set $W = TT^*$ and

$$W_{\alpha\dot\alpha} = P_S^{(2)}(\overline{D}_{\dot\alpha}(W^{-1}D_\alpha W)). \tag{6.22}$$

Comparing Eq. (6.22) with supersymmetric Yang–Mills theory in the chiral representation [22] we see that

$$W_\alpha = \overline{D}_{\dot\alpha}\overline{D}^{\dot\alpha}(W^{-1}D_\alpha W) \tag{6.23}$$

is the curvature in the usual approach to supersymmetric gauge theory if $W \in \mathcal{A}_0$. There it is only this quantity which transforms homogeneously. Whereas it is straightforward to check that in our framework $W_{\alpha\dot\alpha}$ transforms homogeneously under chiral transformations Σ with $\overline{D}\Sigma = 0$:

$$\begin{aligned} T &\longrightarrow T' = \Sigma^* T, \\ W &\longrightarrow W' = \Sigma^* W \Sigma, \\ W_{\alpha\dot\alpha} &\longrightarrow W'_{\alpha\dot\alpha} = \Sigma^{-1} W_{\alpha\dot\alpha}\Sigma. \end{aligned} \tag{6.24}$$

The reason for the homogenous transformation property of $W_{\alpha\dot\alpha}$ is that the inhomogenous term which arises at the level of $Cl_{\mathcal{D}}^{(1,1)}\mathcal{A}$ is in $\mathcal{J}_{\mathcal{D}}^{(1,1)}$.

This allows us to utilize the Wess–Zumino gauge [14] and to rewrite Eq. (6.22) as

$$W_{\alpha\dot\alpha} = \Sigma^{-1} W_{\alpha\dot\alpha}^{WZ} \Sigma \tag{6.25}$$

with

$$W_{\alpha\dot\alpha}^{WZ} = \overline{D}_{\dot\alpha}(\exp - V_{WZ} D_\alpha \exp V_{WZ}) \tag{6.26}$$

and

$$V_{WZ} = -\theta\sigma^\mu\overline{\theta}A_\mu + \mathrm{i}\theta\theta\overline{\theta}\overline{\chi} - \mathrm{i}\overline{\theta}\overline{\theta}\theta\chi + \tfrac{1}{2}\theta\theta\overline{\theta}\overline{\theta}D. \tag{6.27}$$

Thus we infer that the curvature contains a vector field, a Majorana spinor and scalar field modulo chiral gauge transformations.

If it is $W \in \mathcal{A}_v$ then it is $z^\alpha \bar{z}_{\dot\alpha} W_\alpha^{\dot\alpha} \in (\Omega_{\mathcal{D}}^{(1,1)}\mathcal{A})_v$ and hence $z^\alpha \bar{z}_{\dot\alpha} F_\alpha^{\dot\alpha} \in (\Omega_{\mathcal{D}}^{(1,1)}\mathcal{A})_v$. From Eqs. (6.25) and (6.27) we conclude that the curvature is built out of a vector multiplet modulo chiral gauge transformations. Since we want to construct a supersymmetric invariant action we restrict ourselves to the case $T \in \mathcal{A}_v$ and hence $W \in \mathcal{A}_v$. Furthermore, we can write

$$T = i_v(T_0), \quad W = i_v(W_0). \tag{6.28}$$

According to our discussion in the previous section a supersymmetric invariant scalar I for F^2 is given by

$$\begin{aligned} I &= \operatorname{tr}\left(\langle T^*(\overline{D}^{\dot\alpha}(W^{-1}D^\alpha W))(T^{-1})^*, T^*(\overline{D}_{\dot\alpha}(W^{-1}D_\alpha W))(T^{-1})^*\rangle\right) \\ &= \operatorname{tr}\left(\langle \overline{D}^{\dot\alpha}(W^{-1}D^\alpha W), \overline{D}_{\dot\alpha}(W^{-1}D_\alpha W)\rangle\right) \\ &= \operatorname{tr}\int_{\mathcal{M}} (\overline{D}^{\dot\alpha}(W_0^{-1}D^\alpha W_0)\overline{D}_{\dot\alpha}(W_0^{-1}D_\alpha W_0))|_{\theta\theta\bar\theta\bar\theta} \\ &= -\operatorname{tr}\int_{\mathcal{M}} ((W_0^{-1}D^\alpha W_0)\overline{D}\overline{D}(W_0^{-1}D_\alpha W_0))|_{\theta\theta\bar\theta\bar\theta} \\ &= -\operatorname{tr}\int_{\mathcal{M}} (\overline{D}^2(W_0^{-1}D^\alpha W_0)\overline{D}^2(W_0^{-1}D_\alpha W_0))|_{\theta\theta}. \end{aligned} \tag{6.29}$$

Inserting Eqs. (6.27) and (6.25) in Eq. (6.29) we obtain

$$\tfrac{1}{16} I = \operatorname{tr}\int_{\mathcal{M}} -F^{\mu\nu}F_{\mu\nu} - 4\mathrm{i}\chi \not{\nabla}\bar\chi + 2D^2 + \mathrm{i}\varepsilon^{\mu\nu\lambda\rho}F_{\mu\nu}F_{\lambda\rho}, \tag{6.30}$$

which is the action for supersymmetric Yang–Mills theory [14].

7. Conclusions

In this article we have generalized the concept of the spectral triples to algebras which contain both bosonic and fermionic degrees of freedom. The unbounded self-adjoint operator of this triple was constructed out of the spinorial generators of the supersymmetry algebra, i.e. the covariant spinorial derivatives. The construction of the generalized differential algebra out of this spectral triple was discussed in some detail. As a result we obtained that 1-forms of this differential algebra are in the spin $\frac{1}{2}$-representations of the Lorentz-group and, more generally, that n-forms are in the spin $\frac{1}{2}n$-representations. This once more justifies the well-known notion that the covariant spinorial derivatives are the square-roots of the Dirac operator.

For the resulting generalized differential algebra we found that only the finite-dimensional structure of the Grassmann algebra in $\mathcal{A}$ is important, i.e. the generalized differential algebra itself does not contain more information about the underlying bosonic manifold $\mathcal{M}$ than

$C^\infty(\mathcal{M})$. The bosonic part of the algebra becomes important when we consider supersymmetry which can be implemented only on a subalgebra of $\mathcal{A}$ and thus on a subalgebra of $Cl_\mathcal{D}\mathcal{A}$ and $\Omega_\mathcal{D}\mathcal{A}$: the construction of an inner product on $\Omega_\mathcal{D}\mathcal{A}$ which is invariant under supersymmetry transformations involves space–time derivatives.

It is this inner product which marks important deviations from the standard approach to Yang–Mills theory in noncommutative geometry:

(i) The representation of $\Omega\mathcal{A}$ on $\mathcal{H}$ allows to associate to each element in $Cl_\mathcal{D}\mathcal{A}$ an element in $\mathcal{H}$ and therefore the inner product on $\mathcal{H}$ induces an inner product on $Cl_\mathcal{D}\mathcal{A}$. We did not use the Dixmier trace for the definition of the inner product and thus we were not restricted to Euclidean space–time.

(ii) Since the inner product on $Cl_\mathcal{D}\mathcal{A}$ defined via the inner product on $\mathcal{H}$ is indefinite on the subalgebra carrying a supersymmetry representation (and even degenerate on the whole algebra $Cl_\mathcal{D}\mathcal{A}$) we cannot apply the standard procedure for the construction of an inner product on $\Omega_\mathcal{D}\mathcal{A}$. Usually one identifies $\Omega_\mathcal{D}\mathcal{A}$ as the orthogonal complement of the ideal $\mathcal{J}$ in $Cl_\mathcal{D}\mathcal{A}$. This is not possible in our case since the inner product on $Cl_\mathcal{D}\mathcal{A}$ is not positive definite. Therefore we had to use another criterion to map elements of $\Omega_\mathcal{D}\mathcal{A}$ into $Cl_\mathcal{D}\mathcal{A}$. For the subalgebra of $\Omega_\mathcal{D}\mathcal{A}$, which carries a representation of the supersymmetry algebra, we employed the requirement of invariance of the inner product under supersymmetry transformations.

Equipped with this inner product on the generalized differential algebra we followed the standard procedure to construct Yang–Mills theory. However, in our approach to Yang–Mills theory we find as an immediate consequence of the relation between form degree and representation of the Lorentz group that the curvature 2-form is a Lorentz vector and therefore the lowest component in the $\theta, \bar{\theta}$-expansion of the curvature superfield is the vector potential. The curvature superfield does not contain any space–time derivative. Again it is the requirement of invariance under supersymmetry which generates terms containing space–time derivatives in the action for supersymmetric Yang–Mills theory.

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